GAS FLOW THROUGH AN OPENING IN A CHANNEL WALL

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A solution is found for the problem of gas flow through an opening in one of the parallel walls of a channel in which gas is passing. The solution yields, as a limiting case, a solution to the problem of the efflux of gas through a hole in the plane parallel to its direction of flow.

In his solution, the author makes use of a suggestion by Falkovich [1] which allows extension of the Chaplygin gas flow solution [2] to stream problems with several characteristic velocities. The problem dealt with here contains three characteristic velocities.

1. Suppose *AB* and *OF* are channel walls, *DE* is an aperture in a wall, *DM* and *EN* are free surfaces of the jet (Fig. 1) at which the velocity is V_0. Let $2D$ be the channel width, $2d$ the width of the aperture, $2h$ is the width at infinity of the jet flowing out, and v_1 and v_2 are gas velocities at infinitely distant channel sections *AC* and *BF* respectively. We select the center of the aperture *DE* as origin *O*, the x-axis as the line along the channel wall in the main direction of flow and the y-axis as the direction of the jet or stream. Assume that on the streamline *SK*, which branches at point *K*, the stream function $\psi = 0$. If we denote the gas discharge in the stream as q and the discharge through a section of the channel as Q, then $\psi = q$ along *CDM*, and $\psi = -Q$ along the streamline *AB*. Denote the angle between the stream or jet at infinity and the x-axis as α. The gas velocity everywhere is assumed to be subsonic.

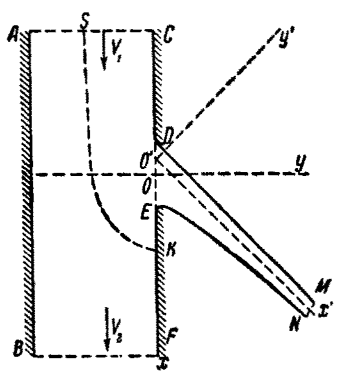

Fig. 1.

Let us put $\tau = v^2/v_{max}^2$ where v is velocity and v_{max} is the maximum flow velocity and θ is the angle of inclination of the velocity vector to the x-axis.

Then in the hodograph plane θ the flow region is represented by a semicircle (Fig. 2).

The boundary conditions are as follows:

$$\begin{aligned}
&\psi = 0 && \text{при } \theta = 0, \quad 0 < \tau < \tau_2 \\
&\psi = -Q && \text{при } \theta = 0, \quad \tau_2 < \tau < \tau_1 \\
&\psi = q && \text{при } \theta = 0, \quad \tau_1 < \tau < \tau_0 \\
&\psi = 0 && \text{при } \theta = \pi, \quad 0 < \tau < \tau_0 \\
&\psi = q && \text{при } \tau = \tau_0, \quad 0 < \theta < m
\end{aligned} \qquad (1.1)$$

$$\psi = 0 \qquad \text{при } \tau = \tau_0, \quad m < \theta < \pi \qquad (1.2)$$

We look for a solution in the following form:

$$\psi_2 = \sum_{n=1}^{\infty} a_n Z_{n/2}(\tau) \sin n\theta \qquad (0 < \tau < \tau_2) \qquad (1.3)$$

$$\psi_1 = -Q\frac{\pi - \theta}{\pi} + \sum_{n=1}^{\infty} \{A_n Z_{n/2}(\tau) + B_n \zeta_{n/2}(\tau)\} \sin \theta \qquad (\tau_2 < \tau < \tau_1) \qquad (1.4)$$

$$\psi_0 = q\frac{\pi - \theta}{\pi} + \sum_{n=1}^{\infty} \{C_n Z_{n/2}(\tau) + D_n \zeta_{n/2}(\tau)\} \sin n\theta \qquad (\tau_1 < \tau < \tau_0) \qquad (1.5)$$

Here $Z_{n/2}(\tau)$ is an integral of the Chaplygin equation [2] regular for $\tau = 0$, whilst $\zeta_{n/2}(\tau)$ is another integral of the same equation linearly independent of $Z_{n/2}$ [3,1]. Essentially, the Wronskian of these integrals will be

$$W(Z_{n/2}, \zeta_{n/2}) = \begin{vmatrix} Z'_{n/2} & \zeta'_{n/2} \\ Z'_{n/2} & \zeta_{n/2} \end{vmatrix} = \frac{n}{2\tau}(1-\tau)^{\beta} \qquad \left(\beta = \frac{1}{\gamma - 1}\right) \qquad (1.6)$$

Here is the polytropic index. The stream function defined by equations (1.3), (1.4), (1.5) satisfies the boundary conditions (1.1). We will now specify that boundary condition (1.2) be satisfied, and that ψ_1 be the analytic continuation of ψ_2 from region $0 < \tau < \tau_2$ into the region $\tau_1 < \tau < \tau_0$, i.e. we require that the following equations hold

$$\begin{aligned}
&\psi_0(\tau_0) = q \qquad (0 < \theta < m) \\
&\psi_0(\tau_0) = 0 \qquad (0 < \theta < \pi)
\end{aligned} \qquad (1.7)$$

$$\begin{aligned}
&\psi_0 = \psi_1, \quad \frac{\partial \psi_0}{\partial \tau} = \frac{\partial \psi_1}{\partial \tau} \quad \text{при } \tau = \tau_1 \\
&\psi_1 = \psi_2, \quad \frac{\partial \psi_1}{\partial \tau} = \frac{\partial \psi_2}{\partial \tau} \quad \text{при } \tau = \tau_2
\end{aligned} \qquad (0 < \theta < \pi) \qquad (1.8)$$

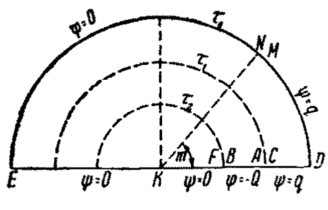

Fig. 2.

If we insert the stream function ψ determined from equations (1.3), (1.4), (1.8) into (1.7) and (1.8) and equate coefficients of sin $n\theta$ we obtain the following system of equations

$$
\begin{aligned}
&C_n Z_{n/2}(\tau_0) + D_n \zeta_{n/2}(\tau_0) = -(2q/\pi n)\cos mn \\
&(C_n - A_n) Z_{n/2}(\tau_1) + (D_n - B_n)\zeta_{n/2}(\tau_1) = -2(q+Q)/\pi n \\
&(C_n - A_r) Z'_{n/2}(\tau_1) + (D_n - B_n)\zeta'_{n/2}(\tau_1) = 0 \\
&(A_n - a_n) Z_{n/2}(\tau_2) + B_n \zeta_{n/2}(\tau_2) = 2Q/\pi n \\
&(A_n - a_n) Z'_{n/2}(\tau_2) + B_n \zeta'_{n/2}(\tau_2) = 0
\end{aligned}
\tag{1.9}
$$

We solve the system (1.9), and making use of relations (1.6), we determine coefficients a_n, A_n, B_n, C_n, D_n.

The stream function ψ is determined likewise. In what follows we will only need to know the function ψ in the region $r_1 < r < r_0$ i.e. ψ_0, which will simply be referred to as ψ. On inserting coefficients C_n, D_n into (1.5) we find

$$\frac{\pi\psi}{2q} = \frac{\pi - \theta}{2} - \sum_{n=1}^{\infty} \frac{1}{n} f_n(\tau)\sin n\theta \tag{1.10}$$

where

$$f_n(\tau) = \cos mn \frac{Z_{n/2}(\tau)}{Z_{n/2}(\tau_0)} - \left\{(1+k)\frac{2\tau_1}{(1-\tau_1)^\beta n}\frac{Z'_{n/2}(\tau_1)}{Z_{n/2}(\tau_0)} - \right. \tag{1.11}$$

$$\left. - k\frac{2\tau_2}{(1-\tau_2)^\beta n}\frac{Z'_{n/2}(\tau_2)}{Z_{n/2}(\tau_0)}\right\}[\zeta_{n/2}(\tau_0) Z_{n/2}(\tau) - Z_{n/2}(\tau_0)\zeta_{n/2}(\tau)] \quad \left(k = \frac{Q}{q}\right)$$

It is easy to see that

$$f_n(\tau_0) = \cos mn \tag{1.12}$$

$$f_n'(\tau_0) = \cos mn \frac{Z'_{n/2}(\tau_0)}{Z_{n/2}(\tau_0)} + k\frac{\tau_2}{\tau_0}\left(\frac{1-\tau_0}{1-\tau_2}\right)^\beta \frac{Z'_{n/2}(\tau_2)}{Z_{n/2}(\tau_0)} - (1+k)\frac{\tau_1}{\tau_0}\left(\frac{1-\tau_0}{1-\tau_1}\right)^\beta \frac{Z'_{n/2}(\tau_1)}{Z_{n/2}(\tau_0)}$$

The last equation will be obtained if we differentiate (1.11) and make use of (1.6).

2. We now introduce a new coordinate system x', y'. For the x'-axis we take the straight line to which both free surfaces of the jet tend (Fig. 1). This straight line intersects the x-axis at O', coordinates $x = a$, $y = 0$. Point O' will be taken as the origin of the new system of coordinates. In the new system we will have

$$\frac{\partial y'}{\partial \theta'} = \frac{1}{v(1-\tau)^\beta}\left[2\tau \frac{\partial \psi}{\partial \tau}\sin\theta' + \frac{\partial \psi}{\partial \theta'}\cos\theta'\right] \tag{2.1}$$

Here $\theta' = \theta - m$. Integrating (2.1) we find

$$\frac{\pi}{2q} y' = \frac{1}{v(1-\tau)^\beta}\left\{-2\tau \sum_{n=1}^{\infty} \frac{1}{n} f_n'(\tau) \int_0^{\theta'} \sin n(\theta'+m)\sin\theta' d\theta' - \right.$$

$$\left. -\frac{1}{2}\int_0^{\theta'} \cos\theta' d\theta' - \sum_{n=1}^{\infty} f_n(\tau)\int_0^{\theta'} \cos n(\theta'+m)\cos\theta' d\theta'\right\} \tag{2.2}$$

Assuming that $\theta = -m$ and $\tau = \tau_0$ in (2.2), we obtain the coordinate $y' = (d + a)\sin m$, at point D. Assuming $\theta' = \pi - m$ and $\tau = \tau_0$ we obtain the coordinate $y' = (a - d)\sin m$ at point E. If we subtract the second relation obtained in this manner from the first, we get

$$\frac{\pi}{2}\frac{d}{h}\sin m = \frac{1}{2}\int_{-m}^{\pi-m} \cos\theta' d\theta' + \sum_{n=1}^{\infty} f_n(\tau_0)\int_{-m}^{\pi-m} \cos n(\theta'+m)\cos\theta' d\theta' +$$

$$+ \sum_{n=1}^{\infty} \frac{2}{n}\tau_0 f_n'(\tau_0)\int_{-m}^{\pi-m} \sin n(\theta'+m)\sin\theta' d\theta' \tag{2.3}$$

Here, it was borne in mind that

$$q = 2hv_0(1-\tau_0)^\beta \tag{2.4}$$

On performing quadrature and taking into account (1.10) we arrive at

$$\frac{\pi}{2}\frac{d}{h} = \pi \operatorname{ctg} m + 1 - 2\sum_{n=1}^{\infty}\frac{\cos 2mn}{4n^2-1} - \sum_{n=1}^{\infty}\frac{4n}{4n^2-1}\frac{\tau_2}{n}\left[\cos 2mn \frac{Z_n'(\tau_0)}{Z_n(\tau_0)} + \right.$$

$$\left. + k\frac{\tau_2}{\tau_0}\left(\frac{1-\tau_0}{1-\tau_2}\right)^\beta \frac{Z_n'(\tau_2)}{Z_n(\tau_0)} - (1+k)\frac{\tau_1}{\tau_0}\left(\frac{1-\tau_0}{1-\tau_1}\right)^\beta \frac{Z_n'(\tau_1)}{Z_n(\tau_0)}\right] \tag{2.5}$$

Notice now that only functions Z_n of integral index remain, because, with functions of the form $Z_{(2k+1)/2}$ the coefficients vanish.

On introducing Chaplygin functions

$$x_n(\tau) = \frac{n}{\tau}\frac{Z_n'(\tau)}{Z_n(\tau)}$$

and bearing in mind that

$$2\sum_{n=1}^{\infty}\frac{\cos 2mn}{4n^2-1} = 1 - \frac{\pi \sin m}{2}$$

we obtain from (2.5)

$$\frac{\pi}{2}\frac{d}{h} = \pi \operatorname{ctg} m + \frac{\pi}{2}\sin m - \tag{2.6}$$

$$- \sum_{n=1}^{\infty} \frac{4n}{4n^2-1}\left\{\cos 2mnx_n(\tau_0) + k\left(\frac{1-\tau_0}{1-\tau_2}\right)^{\beta} \frac{Z_n(\tau_2)}{Z_n(\tau_0)} x_n(\tau_2) - \right.$$

$$\left. - (1+k)\left(\frac{1-\tau_0}{1-\tau_1}\right)^{\beta} \frac{Z_n(\tau_1)}{Z_n(\tau_0)} x_n(\tau_1)\right\}$$

To this expression we should add the equation of continuity

$$Dv_1(1-\tau_1)^{\beta} = Dv_2(1-\tau_2)^{\beta} + hv_0(1-\tau_0)^{\beta} \tag{2.7}$$

Besides

$$k = Q\,/\,q = \frac{v_2(1-\tau_2)^{\beta} D}{v_0(1-\tau_0)^{\beta} h} \tag{2.8}$$

One more equation is obtained from the theorem of momentum conservation:

$$2D(p_1 - p_2) = qv_0\cos m + Qv_2 - 2Dv_1(1-\tau_1)^{\beta} v_1 \tag{2.9}$$

Here p_1 and p_2 are the pressures at the entry and exit of the channel. Taking account of the fact that

$$p = p_0(1-\tau)^{\beta+1}, \qquad p_0 = \frac{v_{\max}^2}{2(\beta+1)} \qquad (\rho_0 = 1)$$

we find from (2.9)

$$\cos m = \frac{1+(2\beta+1)\tau_1}{2(\beta+1)\sqrt{\tau_1\tau_0}} \; \frac{1 - \dfrac{1+(2\beta+1)\tau_2}{1+(2\beta+1)\tau_1}\left(\dfrac{1-\tau_2}{1-\tau_1}\right)^{\beta}}{1 - \left(\dfrac{\tau_2}{\tau_1}\right)^{1/2}\left(\dfrac{1-\tau_2}{1-\tau_1}\right)^{\beta}} \tag{2.10}$$

From expressions (2.6), (2.7), (2.10) h, m, v_2 are determined as functions of v_1, v_0, D and d. From (2.4) we find the discharge q through the hole.

3. If the channel is infinitely wide, $v_1 = v_2$. From (2.10) and (2.6) we find

$$\cos m = \left(\frac{\tau_1}{\tau_0}\right)^{1/2} = \frac{v_1}{v_0} \tag{3.1}$$

$$\frac{\pi}{2}\frac{d}{h} = \pi\operatorname{ctg} m + \frac{\pi}{2}\sin m - \sum_{n=1}^{\infty}\frac{4n}{4n^2-1}\cos 2mnx_n(\tau_0) -$$

$$-\left(\frac{1-\tau_0}{1-\tau_1}\right)^{\beta}\left[\sum_{n=1}^{\infty}\frac{4n}{4n^2-1}\frac{Z_n(\tau_1)}{Z_n(\tau_0)}\,x_n(\tau_1) - 2\sum_{n=1}^{\infty}\frac{4n^2}{4n^2-1}\frac{Z_n(\tau_1)}{Z_n(\tau_0)}\right] \tag{3.2}$$

Putting $v_1 = 0$ in (3.1) and (3.2), we arrive at the case, where gas flows from an infinite vessel and obtain the Chaplygin formula [2]

$$\frac{\pi}{2}\frac{d}{h} = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}4n}{4n^2-1} x_n(\tau_0) \tag{3.3}$$

On replacing τ by v^2/v^2_{max} and going over to the limit where $v_{max} \to \infty$ in (2.6), (2.7), (2.10), we obtain the formulas representing the flow of incompressible fluid from a channel. It is easy to sum the series in this case and the result can be expressed in terms of elementary functions.

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